ISSN 2229-5518

Analytic representation of Transition probability matrix for Exponential Queues with Hyper-exponential service times

Lavanya Gowrishankar[a] , Vidhyacharan Bhaskar[b] and Sundarammal Kesavan[c]
[a,c]*Department of Mathematics,*
S.R.M University, Kattankulathur, Kancheepuram Dt. – 603203, Tamilnadu, INDIA.

Abstract— In this paper we obtain the probability function and moment generating function of an open queuing network node with restriction imposed on their combined distribution. The model is the same as in [1]. The arrival rate of the jobs is taken to be exponential in nature and hence their mixed distribution follows hyper exponential distrib In this paper we obtain the probability function and moment generating function of an open queuing network node with restriction imposed on their combined distribution. The model is the same as in [1]. The arrival rate of the jobs is taken to be exponential in nature and hence their mixed distribution follows hyper exponential distribution. The expression for the transition matrix has been analytically derived.

Index Terms—Exponential distribution, service rate, hyper-exponential distribution, intr arrival distribution, momemt generating function,two dimension random variables,transition probability matrix.

1 INTRODUCTION

1.1 OVERVIEW:

In real world scenario the arrival rate of jobs are not always consistent over a period of time. The arrival rate may follow certain pattern during certain fixed time intervals causing variations in the traffic intensity. To model this kind of situation the Exponential model is more suitable than the Poisson arrival rate model.

In this paper we assume that the arrival rate follows a time dependent exponential distribution. Let the variable X and Y represent two different type of jobs. Let p_1 and p_2 be their respective probabilities. A single queue capable of handling two like jobs as a single entity is considered. The external arrival rate is exponential. The server service rates are hyper exponential. The probability density function and the moment generating function are obtained in this paper. These parameters may be used to calculate other relative measures of the model such as waiting time queue length

Literature Review:

Rapid growth in the communication industry lays emphasis on development of precise and simple models for arrival and service patterns of flowing data. The validity of the model depends mainly on the restrictions based on the arrival pattern., buffer size and the available channels for utility in real world applications. In [1] the authors (we) model an open queuing network node and obtain bounds on two types of job arrivals with restrictions on their combined distribution. The joint density function for all possible combination of job arrivals is analyzed. The mean variance and the square of the coefficient of variation is also calculated. The arrival rate of the jobs are assumed to be exponential. The service rates are analyzed and bond on their probabilities are obtained . In [2] the the authors consider a single server exponential queue with random fluctuations in the intensity of its arrival process. The model does not obey the independence assumption made in Queuing Theory. Necessary and sufficient conditions for the stability or ergodicity of the queuing processes are obtained. Exact performance measures are computed and are compared with the existing results. In this paper “Explicit solutions for queues with Hypo or Hyper exponential service time distribution and application to product form approximations” the authors give a symbolic representation for a rate matrix R for both cases Hypo and Hyper exponential queuing systems. This result is used to address a problem of approximating M/Hypo$_k$ /1 queue by product form model. Rate matrix allows one to specify the approximations for more general models than those that have been previously considered in the literature and with more accuracy. [4] proposes a new approximation for estimating the throughput rate and work-in-process inventory of finite buffered exponential queues in series. The validity of the approximation is established under a wide range of parameters. The cycle time distribution with one single server queue and one exponential multiple server queue is derived in [5].

1.2 Organization of the paper:

Section 2 describes an open node with two segments. The joint density function and combined density function from [1] are used to obtain the probability function and the moment generating function. Also transition states are depicted using a transition probability matrix. Section 3 provides the conclusion.

2. Description:

A single server queue with an arbitrary inter-arrival distribution which is a combination of two different job types X and Y with exponential arrival rates are considered. Let λ_1 and λ_2 be their respective arrival rates. Let t1, t2, …, tn be the inter arrival instants of the jobs. Let x_n $x(t_n\text{-}0)$ represent the number of jobs in the system just before the arrival of the n^{th} job. Z_n denote the number of jobs served

ISSN 2229-5518

during the inter-arrival time (t_n, t_{n+1}) between the n^{th} and $(n+1)^{st}$ job arrival. We represent the state of the system at time t_n. Assuming it is a discrete time system $t_n = nT$ for $n \geq m \geq 0$. Let $p_i(m) = p(x_m = e_i)$ represent the probability that the system goes to e_i at time $t = t_m$. To determine the transition probabilities let y_n denote the number of jobs arriving at the queue during the service time of the n^{th} customer. Then the number of jobs waiting for service at the departure of the $(n+1)^{th}$ customer is $x_n = \begin{cases} x_n + y_{n+1} - 1, & x_n \neq 0 \\ y_{n+1} & x_n = 0 \end{cases}$. The sequence x_n represents a Markov chain and the transition probabilities are obtained using the result $p_{ij} = \begin{cases} p(z_n = i - j + 1) = b_{i-j+1} & i+1 \geq j \geq 1 \\ 0 & j > i+1 \end{cases}$, $p(z_n=j) = b_j$ for j= 0, 1, 2, … represents the probability that j items were served during the inter arrival time τ between the n_{th} and $(n+1)^{th}$ job.. Distribution of τ is hyper-exponential. Using the results that are listed below and $B(z)=\sum_{j=0}^{\infty} b_j z^j$ gives the moment generating function of the random variable z_n using which we try to obtain the transition rate probabilities of the model.

Results used from [1]:

$Z=p_1X+p_2Y$ where X and Y are exponentially distributed with parameters λ_1 and λ_2 then $f_{XY}(x,y)= \lambda_1 \lambda_2 e^{-(\lambda_1 x+ \lambda_2 y)}$, x>0, y>0,. Introducing auxillary random variable $W=p_2Y$ we have $p_1X = Z-W$ and

$$p_2 Y = W, \quad K = \frac{1}{p_1 p_2},$$

$$f_{ZW}(z,w) = |J| f_{XY}(x,y)$$

$$= K\lambda_1\lambda_2 e^{-\left(\frac{\lambda_1}{p_1}(z-w)+\frac{\lambda_2}{p_2}(w)\right)}, z, w, \geq 0.$$

$$\text{The pdf of } z = f_z(z) = K\lambda_1\lambda_2 e^{-\left(\lambda_1'(w)\right)} z \geq 0,$$

$$\lambda_1' = \frac{\lambda_1}{p_1}, \lambda_2' = \frac{\lambda_2}{p_2}.$$

Theorem: For a hyper-exponential distribution with two states, the h_2 function has three parameters (p_1, μ_1, μ_2) with the following representation.

$P=[p_1,p_2]$, $B = \begin{bmatrix} \mu_1 & 0 \\ 0 & \mu_2 \end{bmatrix}, T = \begin{bmatrix} T_1 & 0 \\ 0 & T_2 \end{bmatrix} \text{where } T_i = \frac{1}{\mu_i}.$

If the first three moment are known, one can find the bounds for p_1 and p_2 under the following conditions

$$\gamma = \frac{c_v^2 - 1}{2}, p_1 + p_2 = 1,$$

$$\text{For } T_2 < T_1 \begin{cases} T_1 = \bar{x}\left[1 + \sqrt{\frac{p_2\gamma}{p_1}}\right] \\ T_2 = \bar{x}\left[1 - \sqrt{\frac{p_1\gamma}{p_2}}\right] \end{cases}, \text{only if } p_1\gamma < p_2$$

The service rates T_1 and T_2 are hyper-exponential if $c_v^2 > 1.$

$P(z_j=n) = b_n = \int_0^\infty P(z_j = n) f_\tau(\tau) d\tau$ where b_n represents the probability that n jobs were processed during two consecutive time instants that is during the inter arrival time τ. Here,

$P(z_j=n)$ is $f_z(z) = K\lambda_1\lambda_2 e^{-\left(\lambda_1'(w)\right)} z \geq 0, \lambda_1' = \frac{\lambda_1}{p_1}, \lambda_2' = \frac{\lambda_2}{p_2}.$

Also the distribution of τ $A(\tau)$ is hyper-exponential in nature . Clearly the pdf for an hyper-exponentialdistribution is

$$\sum_{i=1}^{n} \frac{p_i}{\alpha_i} e^{\frac{-\tau}{\alpha_i}} \quad \tau > 0.$$

Therefore

$$b_n = \int_0^\infty \sum_{i=1}^{n} \frac{p_i}{\alpha_i} e^{-\frac{\tau}{\alpha_i}} \left(\frac{k\lambda_1\lambda_2}{\lambda_2' - \lambda_1'}\right) e^{-\lambda_1'\tau} d\tau$$

$$= K \int_0^\infty \sum_{i=1}^{n} e^{-\frac{\tau}{\alpha_i}} e^{-\lambda_1'\tau} d\tau \text{ where we consider } \frac{p_i}{\alpha_i} = a \text{ for all } i,$$

$$K = \frac{ak\lambda_1\lambda_2}{\lambda_2' - \lambda_1'}$$

$$\text{then } b_n = K \sum_{i=1}^{n} \int_0^\infty e^{-\left(\frac{1}{\alpha_i}+\lambda_1'\right)\tau} d\tau\tau$$

$$b_n = K \sum_{i=1}^{n} \frac{1}{\frac{1}{\alpha_i} + \lambda_1'} = K'.$$

Therefore the moment generating function of the random variable z_j is found to be $B(z) = \sum_{i=0}^{\infty} b_i z^i = K' \sum_{i=0}^{\infty} z^i = \frac{K'}{1-z}.$

Clearly p_{ij} represents the transition probabilities of the sequence $\{x_n\}$ which represents a Markov chain using the relation $x_{n+1}=x_n+1-z_n$, $x_n \geq 0$ and $z_n \leq x_{n+1}$. Hence p_{ij} is valid only for $j \geq 1$to obtain p_{i0}

$$\sum_{j=0}^{i+1} p_{ij} = 1 \quad for \ i = 0,1,...$$

$$p_{i0} = 1 - \sum_{j=1}^{i+1} p_{ij} = 1 - \sum_{j=1}^{i+1} b_{i+j}$$

$$p_{i0} = 1 - \sum_{k=0}^{i} b_k = c_i$$

$$p_{10} = 1 - \sum_{k=0}^{1} b_k = 1 - \left[\frac{1}{\frac{1}{\alpha_0} + \lambda_1'} + \frac{1}{\frac{1}{\alpha_1} + \lambda_1'} \right]$$

$$= 1 - \left[\frac{\alpha_0}{1 + \alpha_0 \lambda_1'} + \frac{\alpha_1}{1 + \alpha_1 \lambda_1'} \right]$$

$$p_{20} = 1 - \sum_{k=0}^{2} b_k = 1 - \left[\frac{1}{\frac{1}{\alpha_0} + \lambda_1'} + \frac{1}{\frac{1}{\alpha_1} + \lambda_1'} + \frac{1}{\frac{1}{\alpha_2} + \lambda_1'} \right]$$

$$c_2 = 1 - \left[\frac{\alpha_0}{1 + \alpha_0 \lambda_1'} + \frac{\alpha_1}{1 + \alpha_1 \lambda_1'} + \frac{\alpha_2}{1 + \alpha_2 \lambda_1'} \right]$$

$$c_2 = p_{20} = p_{10} + \frac{\alpha_2}{1 + \alpha_2 \lambda_1'}$$

$$c_3 = p_{30} = p_{20} + \frac{\alpha_3}{1 + \alpha_3 \lambda_1'}$$

$$c_k = p_{k0} = p_{(k-1)0} + \frac{\alpha_k}{1 + \alpha_k \lambda_1'}$$

Similarly,

$$b_i = K \sum_{i=1}^{n} \frac{1}{\frac{1}{\alpha_i} + \lambda_1'}$$

$$P_{ij} = b_{i-j+1}$$

$P_{12} = b_0$, $p_{22} = b_1$ and so on. Treating the case that atleast one job is processed in time τ

$$b_1 = \int_0^{\infty} \frac{p_1}{\alpha_1} e^{-\frac{\tau}{\alpha_1}} \left(\frac{k \lambda_1 \lambda_2}{\lambda_2' - \lambda_1'} \right) e^{-\lambda_1' \tau} d\tau$$

$$= K \left(\frac{\alpha_1}{1 + \alpha_1 \lambda_1'} \right)$$

$$b_2 = b_1 + K \left(\frac{\alpha_2}{1 + \alpha_2 \lambda_1'} \right)$$

$$b_k = b_{(k-1)} + K \left(\frac{\alpha_k}{1 + \alpha_k \lambda_1'} \right)$$

The transition probability matrix is expressed as

$$P = \begin{bmatrix} c_0 & b_0 & 0 & 0 & . & . & . & . & .. & 0 \\ c_1 & b_1 & b_0 & . & . & . & . & . & . & 0 \\ c_2 & b_2 & b_1 & b_0 & . & . & . & . & . & 0 \\ c_3 & b_3 & b_2 & b_2 & b_1 & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 \\ . & . & . & . & . & . & . & . & . & 0 \\ c_k & b_k & b_{k-1} & . & . & . & . & . & b_0 & 0 \\ . & . & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & . & . \end{bmatrix}$$

Conclusion: In this paper the probability that n jobs are processed in time interval (n, n+1) is found. The transition probabilities are derived analticaly keeping the constraint that the service rate is hyper-exponential. Further numerical results are to be executed and using these expressions for moment generating functions other relative measures for the model could be computed.

References:

[1] Lavanya Gowrishankar$_a$, Vidhyacharan Bhaskar pand Sundarammal Kesava, "Treating two like jobs as a single entity - A two server model," ICMS 2014, Sathyabama University, pp.772-775, 17-19th July.2014.

[2] Catherine Rosenberg, Ravi Mazumdar, Leonard Klienrock,"On the Analysis of exponential queuing systems with randomly changing arrival rates : Stability conditions and finite buffer schemes with a resume level," Performance Evaluation, vol.11, Iss.4, pp.283-292, Nov. 1990,.

[3] Andrea Martin, Samuel Rota Bulb, "Explicit solutions for queues with Hypo-or-Hyper exponential service time distribution and application to product form pproximations," Performance Evaluation, vol. 81, pp.1-19, Nov. 24.

[4] Mark C. Springer, " A decomposition approximation for finite-buffered flow lines of exponential queues," European Journal of Operations Research, vol.74, Iss.1, pp. 75-110, 7 April.1994.

[5] Tze Fen Li, "Statistics and Probability Letters, vol.6, Iss.2, pp. 77-81, Nov.1987.

[6] Sheldon Ross, "Introduction to Probability Models," Chapter 5, pp.277-356,11th Edition, 2014.